McBits: Fast code-based cryptography

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Joint work with Daniel Bernstein, Tung Chou

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Introduction: the bigger context

Public-key encryption

- Alice generates a key pair (sk, pk), publishes pk, keeps sk secret
- Bob takes some message M and pk and computes an **ciphertext** C, sends C to Alice
- Alice uses sk to decrypt C and obtain M

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Implementation targets

- Secure
- Fast
- (Small, low energy, low-power,...)

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- *Constant-time* software avoids such timing leaks:
  - No secret branch conditions
  - No memory access with secret address (*cache timing*)

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- This talk: focus on high throughput for servers
- Target micro-architecture: Intel Sandy Bridge/Ivy Bridge
- Techniques also interesting for other (micro-)architectures

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- Highest arithmetic throughput, example (Sandy Bridge):
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- Low performance for branches, independent vector-element loads
- Synergie between efficient vectorization and timing-attack protection

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- Other views on bitslicing:
  - Computations on a transposition of data
  - Simulation of hardware implementations in software
- Needs large degree of data-level parallelism (e.g., $128\times$)
- Size of active data set increases massively (e.g., $128\times$)
- Typical consequence: more loads and stores (that easily become the performance bottleneck)

A code-based cryptosystem

System parameters

- Integers m, n, t, k, such that
  - $n \leq 2^m$
  - $k = n - mt$
  - $t \geq 2$

Example

- $m = 12$,
  $n = 4096$
  $k = 3604$
  $t = 41$

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- An $(s + a)$-bit-output hash function H

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- $H =$ SHA-512

Key generation

Secret key

- A random sequence $(\alpha_1, \ldots, \alpha_n)$ of distinct elements in $\mathbb{F}_{2^m}$
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$$\begin{pmatrix} 1/g(\alpha_1) & 1/g(\alpha_2) & \cdots & 1/g(\alpha_n) \\ \alpha_1/g(\alpha_1) & \alpha_2/g(\alpha_2) & \cdots & \alpha_n/g(\alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{t-1}/g(\alpha_1) & \alpha_2^{t-1}/g(\alpha_2) & \cdots & \alpha_n^{t-1}/g(\alpha_n) \end{pmatrix} \in \mathbb{F}_{2^m}^{t \times n}$$

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- Obtain a matrix $H_{sec} \in \mathbb{F}_2^{mt \times n}$

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- H_{sec} is a *secret* parity-check matrix of the Goppa code $\Gamma = \Gamma_2(\alpha_1, \ldots, \alpha_n, g)$
- The secret key is $(\alpha_1, \ldots, \alpha_n, g)$

Key generation

Public key

- Perform Gaussian elimination on H_{sec} to obtain a matrix H_{pub} whose left $tm \times tm$ submatrix is the identity matrix
- H_{pub} is a *public* parity-check matrix for Γ
- The public key is H_{pub}

Encryption

- Generate a random weight-t vector $e \in \mathbb{F}_2^n$
- Compute $w = H_{pub}e$
- Compute $H(e)$ to obtain an $(s+a)$-bit string (k_{enc}, k_{auth})
- Encrypt the message M with the stream cipher S under key k_{enc} to obtain ciphertext C
- Compute authentication tag a on C using A with key k_{auth}
- Send (a, w, C)

Decryption

- Receive (a, w, C)
- Decode w to obtain weight-t string e
- Hash e with H to obtain (k_{enc}, k_{auth})
- Verify that a is a valid authentication tag on C using A with k_{auth}
- Use S with k_{enc} to decrypt and obtain M

Software implementation, first considerations

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- Some hassle to make constant-time, but possible

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Decryption

- Decryption is mainly decoding, lots of operations in $\mathbb{F}_{2^m}$
- Decryption has to run in constant time!
- Obviously, decoding of w is the interesting part

A closer look at decoding

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- All these computation work on medium-size polynomials over $\mathbb{F}_{2^m}$
- Let's now fix the example parameters from above $(n = 2^m = 4096, t = 41)$

Representing elements of $\mathbb{F}_{2^m}$

Option I

- Use 16-bit integer values (`unsigned short`)
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  - Squaring uses the same algorithm as multiplication

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- Needs many parallel computations, obtain parallelism from independent decryption operations
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- Multiplication is easily constant time, but is it fast?
- How about squaring, can it be faster?

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\begin{aligned}
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= \quad & (1 - x^n)(a_0 b_0 - x^n a_1 b_1) + x^n (a_0 + a_1)(b_0 + b_1)
\end{aligned}
$$

- Refined Karatsuba uses $M_{2n} = 3M_n + 7n - 3$ instead of $M_{2n} = 3M_n + 8n - 4$ bit operations
- For details see Bernstein, "Batch binary Edwards", Crypto 2009

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- In the following: High-level algorithms that drastically reduce the number of multiplications

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- Same operation count but different structure
- Berlekamp's trace algorithm: not constant time

Multipoint evaluation via FFT

- Evaluate a polynomial $f = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$ at all n-th roots of unity
- Divide-and-conquer approach
  - Write polynomial f as $f_0(x^2) + x f_1(x^2)$

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- Gao, Mateer 2010: Much faster additive FFT

Gao-Mateer additive FFT

- Evaluate a polynomial $f = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$ on a size-n $\mathbb{F}_2$-linear space S
- Idea: Write polynomial f as $f_0(x^2 + x) + xf_1(x^2 + x)$

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Gao-Mateer additive FFT

- Evaluate a polynomial $f = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ on a size-n $\mathbb{F}_2$-linear space S
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- Again: apply the idea recursively
- Our paper: generalize the idea to small-degree f
  - Recursion can stop much earlier
  - Various speedups at the end of the recursion

Syndrome computation, the classical way

- Receive n-bit input word, scale bits by Goppa constants
- Apply linear map

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{2t-1} & \alpha_2^{2t-1} & \cdots & \alpha_n^{2t-1} \end{pmatrix}$$

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- Can precompute matrix mapping bits to syndrome
- Yields pretty large secret key, larger than L1 cache

Another look at syndrome computation

Look at the syndrome-computation map again:

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{2t-1} & \alpha_2^{2t-1} & \cdots & \alpha_n^{2t-1} \end{pmatrix}$$

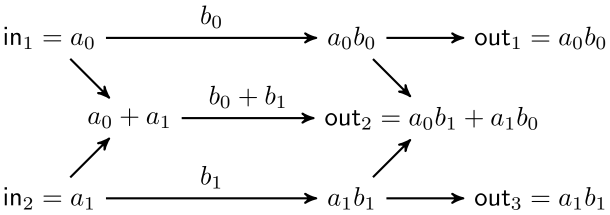
Consider the linear map $M^\intercal$:

$$\begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{2t-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{2t-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \cdots & \alpha_n^{2t-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_t \end{pmatrix} = \begin{pmatrix} v_1 + v_2\alpha_1 + \cdots + v_t\alpha_1^{2t-1} \\ v_1 + v_2\alpha_2 + \cdots + v_t\alpha_2^{2t-1} \\ \vdots \\ v_1 + v_2\alpha_n + \cdots + v_t\alpha_n^{2t-1} \end{pmatrix} = \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ \vdots \\ f(\alpha_n) \end{pmatrix}$$

- This transposed linear map is actually doing multipoint evaluation
- Syndrome computation is a transposed multipoint evaluation

Transposing linear algorithms

- A linear map: $a_0, a_1 \to a_0 b_0, a_0 b_1 + a_1 b_0, a_1 b_1$



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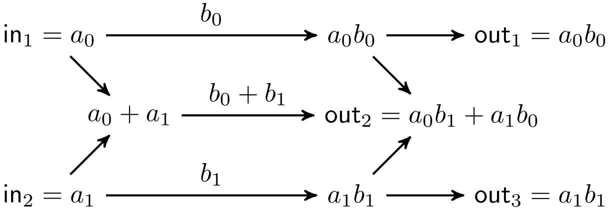
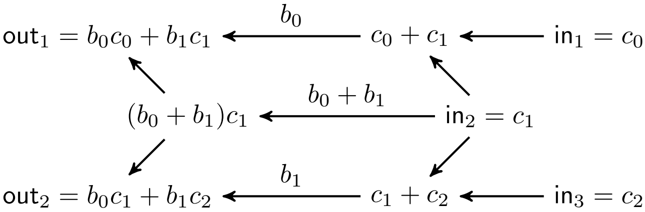


- Reversing the edges: $c_0, c_1, c_2 \to b_0c_0 + b_1c_1, b_0c_1 + b_1c_2$



What did we just do?

- The original linear map:

$$\begin{pmatrix} a_0b_0 \\ a_0b_1 + a_1b_0 \\ a_1b_1 \end{pmatrix} = \begin{pmatrix} b_0 & 0 \\ b_1 & b_0 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

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- This is called the *transposition principle*
- Preserves number of multiplications
- References: Fiduccia 1972, Bordewijk 1956, Lupanov 1956

Transposing the additive FFT

The naive approach

- Idea: Compute syndrome by transposing the additive FFT
- Start with additive FFT program (sequence of additions and constant multiplications)
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  - At $m = 13$ or $m = 14$ gcc runs out of memory
  - Can use better register allocators, but the program is still huge

Transposing the additive FFT

A better approach

- Analyze structure of additive FFT A: B, A_1, A_2, C
- A_1, A_2 are recursive calls

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- Analyze structure of additive FFT A: B, A_1, A_2, C
- A_1, A_2 are recursive calls
- Transposition has structure C^T, A_2^T, A_1^T, B^T
- Use recursive calls to reduce code size

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- Solution: sorting networks

Sorting networks

A *sorting network* sorts an array S of elements by using a sequence of *comparators*.

- A comparator can be expressed by a pair of indices (i, j).
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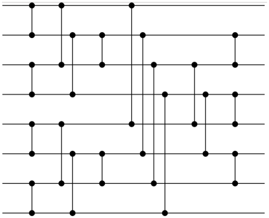
- Efficient sorting network: Batcher sort (Batcher, 1968)



Batcher sorting network for sorting 8 elements
http://en.wikipedia.org/wiki/Batcher%27s_sort

Permuting by sorting

Example

Computing b_3, b_2, b_1 from b_1, b_2, b_3 can be done by sorting the key-value pairs $(3, b_1), (2, b_2), (1, b_3)$ the output is $(1, b_3), (2, b_2), (3, b_1)$

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- Possibly better than Batcher sort: Beneš permutation network (work in progress)

Results

Throughput cycles on Ivy Bridge

- Input secret permutation: 8622
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- Total: **60493**

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- All computations with full timing-attack protection!

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Public-key decryption speeds from eBATS

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Diffie-Hellman shared-secret speeds from eBATS

- `gls254`: 77468 cycles
- `kumfp127g` 116944 cycles
- `curve25519`: 182632 cycles

References

- Daniel J. Bernstein, Tung Chou, and Peter Schwabe. *McBits: fast constant-time code-based cryptography.*, CHES 2013. http://cryptojedi.org/papers/#mcbits

- Software will be online (public domain), for example, at http://cryptojedi.org/crypto/#mcbits