Implementing post-quantum cryptography

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Part I: How to make software secure

Timing Attacks

General idea of those attacks

- Secret data has influence on timing of software
- Attacker measures timing
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- Unlike other side-channel attacks, they work remotely:
  - Some need to run attack code in parallel to the target software
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  - Attacker can log in remotely (ssh)
  - Some attacks work by measuring network delays
  - Attacker does not even need an account on the target machine
- Can't protect against timing attacks by locking a room
- This talk: don't consider "local" side-channel attacks

Problem No. 1

```
if(secret)
{
  do_A();
}
else
{
  do_B();
}
```

Examples

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- Byte-array (tag) comparison:

  "if $a[i] \neq b[i]$: return"

- Sorting and permuting:

  "if $a < b$: branch into subroutine"

Eliminating branches

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- Can expand s to all-one/all-zero mask and use XOR instead of addition, AND instead of multiplication
- For very fast A and B this can even be faster

Problem No. 2

`table[secret]`

Timing leakage part II

$T[0] \dots T[15]$
$T[16] \dots T[31]$
$T[32] \dots T[47]$
$T[48] \dots T[63]$
$T[64] \dots T[79]$
$T[80] \dots T[95]$
$T[96] \dots T[111]$
$T[112] \dots T[127]$
$T[128] \dots T[143]$
$T[144] \dots T[159]$
$T[160] \dots T[175]$
$T[176] \dots T[191]$
$T[192] \dots T[207]$
$T[208] \dots T[223]$
$T[224] \dots T[239]$
$T[240] \dots T[255]$

- Consider lookup table of 32-bit integers
- *Cache lines* have 64 bytes
- Crypto and the attacker's program run on the same CPU
- Tables are in cache

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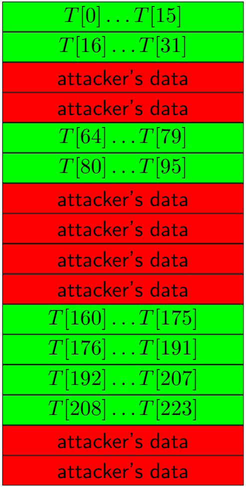

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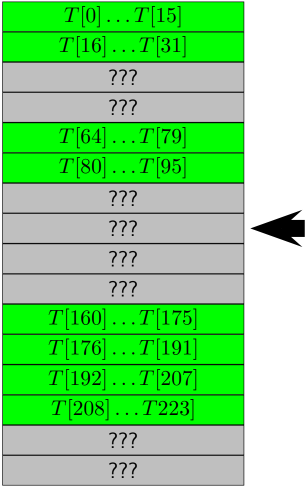

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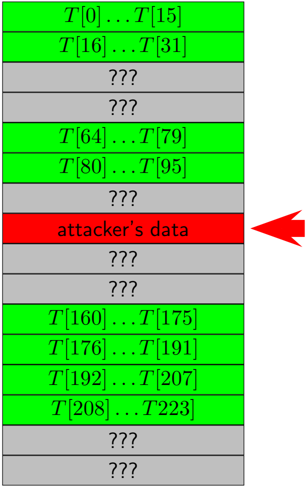

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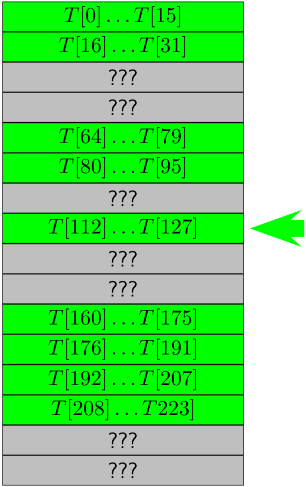

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- Crypto and the attacker's program run on the same CPU
- Tables are in cache
- The attacker's program replaces some cache lines
- Crypto continues, loads from table again
- Attacker loads his data:
  - Fast: cache hit (crypto did not just load from this line)
  - Slow: cache miss (crypto just loaded from this line)

The general case

Loads from and stores to addresses that depend on secret data leak secret data.

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- Yarom, Genkin, Heninger: CacheBleed attack *"is able to recover both 2048-bit and 4096-bit RSA secret keys from OpenSSL 1.0.2f running on Intel Sandy Bridge processors after observing only 16,000 secret-key operations (decryption, signatures)."*

Countermeasure

```
uint32_t table[TABLE_LENGTH];

uint32_t lookup(size_t pos)
{
  size_t i;
  int b;
  uint32_t r = table[0];
  for(i=1;i<TABLE_LENGTH;i++)
  {
    b = (i == pos);
    cmov(&r, &table[i], b); // See "eliminating branches"
  }
  return r;
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  for(i=1;i<TABLE_LENGTH;i++)
  {
    b = (i == pos); /* DON'T! Compiler may do funny things! */
    cmov(&r, &table[i], b);
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  for(i=1;i<TABLE_LENGTH;i++)
  {
    b = isequal(i, pos);
    cmov(&r, &table[i], b);
  }
  return r;
}
```

Countermeasure, part 2

```
int isequal(uint32_t a, uint32_t b)
{
  size_t i; uint32_t r = 0;
  unsigned char *ta = (unsigned char *)&a;
  unsigned char *tb = (unsigned char *)&b;
  for(i=0;i<sizeof(uint32_t);i++)
  {
    r |= (ta[i] ^ tb[i]);
  }
  r = (-r) >> 31;
  return (int)(1-r);
}
```

Part II: How to make software fast

Vector computations

Scalar computation

- Load 32-bit integer a
- Load 32-bit integer b
- Perform addition $c \leftarrow a + b$
- Store 32-bit integer c

Vectorized computation

- Load 4 consecutive 32-bit integers (a_0, a_1, a_2, a_3)
- Load 4 consecutive 32-bit integers (b_0, b_1, b_2, b_3)
- Perform addition $(c_0, c_1, c_2, c_3) \leftarrow (a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3)$
- Store 128-bit vector (c_0, c_1, c_2, c_3)

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- Vector instructions available on most "large" processors
- Instructions for vectors of bytes, integers, floats...

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  - 256-bit store throughput: 1 per cycle
- **Vector instructions are almost as fast as scalar instructions but do $8\times$ the work**
- Situation on other architectures/microarchitectures is similar
- Reason: cheap way to increase arithmetic throughput (less decoding, address computation, etc.)

Take-home message

"Big multipliers are pre-quantum,
vectorization is post-quantum"

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  - Typically ignore some results
- Reason: reuse coefficients of $\mathbf{A}$ in cache

Structured lattices

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r_0 &= f_0 g_0 \\
r_1 &= f_0 g_1 + f_1 g_0 \\
r_2 &= f_0 g_2 + f_1 g_1 + f_2 g_0 \\
r_3 &= f_0 g_3 + f_1 g_2 + f_2 g_1 + f_3 g_0 \\
r_4 &= f_1 g_3 + f_2 g_2 + f_3 g_1 \\
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- Can easily load (f_0, f_1, f_2, f_3) and (g_0, g_1, g_2, g_3)
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- Multiply, obtain $(f_0g_0, f_1g_1, f_2g_2, f_3g_3)$
- And now what?
- Looks like we need to *shuffle* a lot!

Karatsuba and Toom

- Our polynomials have many more coefficients (say, 256–1024)
- Idea: use Karatsuba's trick:
  - consider $n = 2k$-coefficient polynomials f and g
  - Split multiplication $f \cdot g$ into 3 half-size multiplications

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& (f_\ell + X^k f_h) \cdot (g_\ell + X^k g_h) \\
= \; & f_\ell g_\ell + X^k (f_\ell g_h + f_h g_\ell) + X^n f_h g_h \\
= \; & f_\ell g_\ell + X^k ((f_\ell + f_h)(g_\ell + g_h) - f_\ell g_\ell - f_h g_h) + X^n f_h g_h
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- Apply recursively to obtain 9 quarter-size multiplications, 27 eighth-size multiplications etc.

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$$

- Apply recursively to obtain 9 quarter-size multiplications, 27 eighth-size multiplications etc.
- Generalization: Toom-Cook. Obtain, e.g., 5 third-size multiplications
- Split into sufficiently many "small" multiplications, vectorize across those

Transposing/Interleaving

- Small example: compute $a \cdot b$, $c \cdot d$, $e \cdot f$, $g \cdot h$
- Each factor with 3 coefficients, e.g., $a = a_0 + a_1 X + a_2 X^2$

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- Solution: transpose data matrix (or interleave words):

```
a0, c0, e0, h0, a1, c1, e1,..., f2, g2
```

Two applications of Karatsuba/Toom

Streamlined NTRU Prime 4591^{761}

- Multiply in the ring $\mathcal{R} = \mathbb{Z}_{4591}[X]/(X^{761} - X - 1)$
- Pad input polynomial to 768 coefficients
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NTRU-HRSS-KEM

- Multiply in the ring $\mathcal{R} = \mathbb{Z}_{8192}[X]/(X^{701} - 1)$
- Use Toom-Cook to split into 7 quarter-size, then 2 levels of Karatsuba
- Obtain 63 multiplications of 44-coefficient polynomials
- $11\,722$ Haswell cycles for multiplication in $\mathcal{R}$

We can do better: NTTs

- Many LWE/MLWE systems use very specific parameters:
  - Work in polynomial ring $\mathcal{R} = \mathbb{Z}_q[X]/(X^n + 1)$
  - Choose n a power of 2
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- Big advantage: fast *negacyclic number-theoretic transform*
- Given $g \in \mathcal{R}$, n-th primitive root of unity ω and $\psi = \sqrt{\omega}$, compute

$$\text{NTT}(g) = \hat{g} = \sum_{i=0}^{n-1} \hat{g}_i X^i, \text{ with}$$

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- NTT^{-1} is essentially the same computation as NTT

Zooming into the NTT

- FFT in a finite field
- Evaluate polynomial $f = f_0 + f_1 X + \cdots + f_{n-1} X^{n-1}$ at all n-th roots of unity
- Divide-and-conquer approach
  - Write polynomial f as $f_0(X^2) + X f_1(X^2)$

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  - Same for f_1
- Apply recursively through $\log n$ levels

Vectorizing the NTT

- First thing to do: replace recursion by iteration
- Loop over $\log n$ levels with $n/2$ "butterflies" each
- Butterfly on level k:
  - Pick up f_i and f_{i+2^k}
  - Multiply f_{i+2^k} by a power of ω to obtain t
  - Compute $f_{i+2^k} \leftarrow a_i - t$
  - Compute $f_i \leftarrow a_i + t$
- All $n/2$ butterflies on one level are independent
- Vectorize across those butterflies

Vectorized NTT results

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- Seiler, 2018:
  - 2784 Haswell cycles ($n = 1024$, 14-bit q)
  - 460 Haswell cycles ($n = 256$, 13-bit q)
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How about hashing?

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- Consequence: consider designing with parallel hash/XOF calls!

PQCRYPTO $\neq$ Lattices

- So far we've looked at lattices, how about other PQCRYPTO?
- Code-based crypto (and some $\mathcal{MQ}$-based crypto) need binary-field arithmetic
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- Obvious question: can vector operations help?

Bitslicing

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- Consider now vectors of bits
- Perform arithmetic on those vectors using `XOR`, `AND`, `OR`
- "Simulate hardware implemenations in software"
- Technique was introduced by Biham in 1997 for DES
- Bitslicing works for every algorithm
- *Efficient* bitslicing needs a huge amount of data-level parallelism

Bitslicing binary polynomials

4-coefficient binary polynomials

$(a_3x^3 + a_2x^2 + a_1x + a_0)$, with $a_i \in \{0, 1\}$

4-coefficient bitsliced binary polynomials

```
typedef unsigned char poly4; /* 4 coefficients in the low 4 bits */
typedef unsigned long long poly4x64[4];

void poly4_bitslice(poly4x64 r, const poly4 f[64])
{
  int i,j;
  for(i=0;i<4;i++)
  {
    r[i] = 0;
    for(j=0;j<64;j++)
      r[i] |= (unsigned long long)(1 & (f[j] >> i))<<j;
  }
}
```

Bitsliced binary-polynomial multiplication

```
typedef unsigned long long poly4x64[4];
typedef unsigned long long poly7x64[7];

void poly4x64_mul(poly7x64 r, const poly4x64 f, const poly4x64 g)
{
  r[0] = f[0] & g[0];
  r[1] = (f[0] & g[1]) ^ (f[1] & g[0]);
  r[2] = (f[0] & g[2]) ^ (f[1] & g[1]) ^ (f[2] & g[0]);
  r[3] = (f[0] & g[3]) ^ (f[1] & g[2]) ^ (f[2] & g[1]) ^ (f[3] & g[0]);
  r[4] = (f[1] & g[3]) ^ (f[2] & g[2]) ^ (f[3] & g[1]);
  r[5] = (f[2] & g[3]) ^ (f[3] & g[2]);
  r[6] = (f[3] & g[3]);
}
```

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  - Reason: Need 256 independent decodings for parallelism
- Chou, CHES 2017: use *internal* parallelism
  - Target even higher security (297 bits pre-quantum)
  - Does *not* require independent decryptions
  - Even faster, even when considering throughput

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- $\mathbb{F}_2/\mathbb{F}_4$: Use bitslicing
- $\mathbb{F}_{16}/\mathbb{F}_{256}$: Use vector-permute instructions for table lookups
- For $\mathbb{F}_{256}$ use tower-field arithmetic on top of $\mathbb{F}_{16}$

Recent $\mathcal{MQ}$ results

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  - Bernstein, Hopwood, Hülsing, Lange, Niederhagen, Papachristodoulou, Schneider, Schwabe, Wilcox-O'Hearn, 2015: Optimize SPHINCS
  - Vectorize also Merkle-tree hashes inside HORST computation
  - ≈ 52 Mio cycles for signing on Haswell

Additional benefits

Two things very inefficient to vectorize

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- More importantly: eliminates most notorious timing side channels!
- Efficient vectorized implementations are often also “constant-time”

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