Engineering Cryptographic Software

Elliptic-Curve Arithmetic

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January 2026

Where were we?

- For scalar multiplication, we assumed a group G
  - of finite order ℓ,
  - that is commutative (Abelian),
  - that is cyclic with generator P, and
  - in which the **discrete-logarithm problem is hard**.

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- Today: make this group concrete

Fields

Definition

A set S together with two operations $(+, \cdot)$ is called a *field* $K = (S, +, \cdot)$ if

- $(S, +)$ is an Abelian group
- $(S \setminus \{0\}, \cdot)$ is an Abelian group, where 0 is the neutral element of $(S, +)$
- For all $a, b, c \in S$ it holds that $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity)

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  - If char$(K) = p \neq 0$, then p is prime

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- The smallest field is $\{0, 1\}$ with addition and multiplication modulo 2
  - Addition is XOR
  - Multiplication is AND

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Typical view on elliptic curves

Definition

Let K be a field with char$(K) \notin \{2, 3\}$ and let $a, b \in K$. Then the following equation defines an elliptic curve E:

$$E : y^2 = x^3 + ax + b,$$

if the discriminant $\Delta = -64a^3 - 432b^2$ of E is not equal to zero. This equation is called the *short Weierstrass form* of an elliptic curve.

Rational points

Setup for cryptography

- Choose $K = \mathbb{F}_q$
- Consider the set of $\mathbb{F}_q$-rational points:

$$E(\mathbb{F}_q) = \{(x, y) \in \mathbb{F}_q \times \mathbb{F}_q : y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$$

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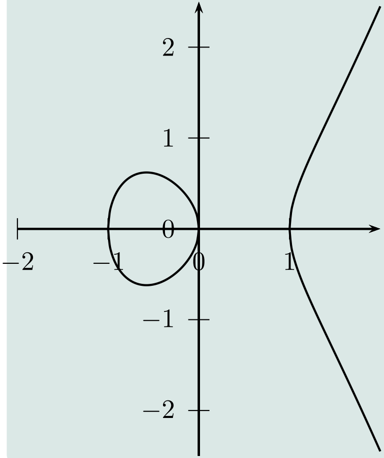
- Order of this group: $|E(\mathbb{F}_q)| \approx |\mathbb{F}_q|$

Graph of E over $\mathbb{R}$



Addition of points

- Add points
  $P = (-0,9; -0,4135)$ and
  $Q = (-0,1; 0,3146)$

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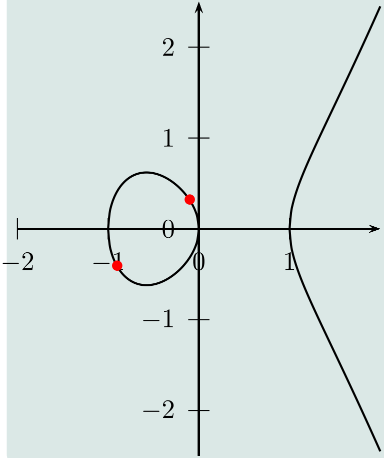

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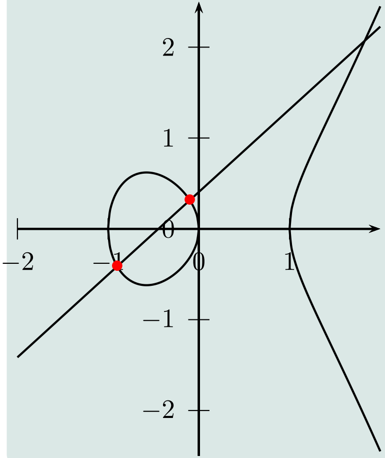

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- Determine third intersection $T = (x_T, y_T)$ with the elliptic curve

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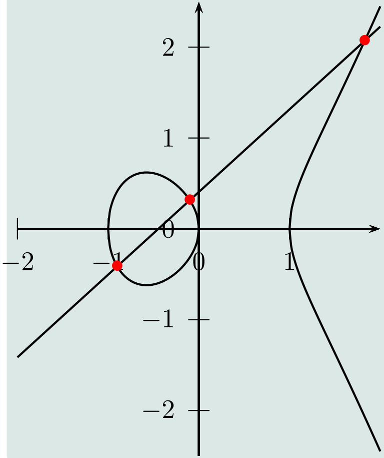

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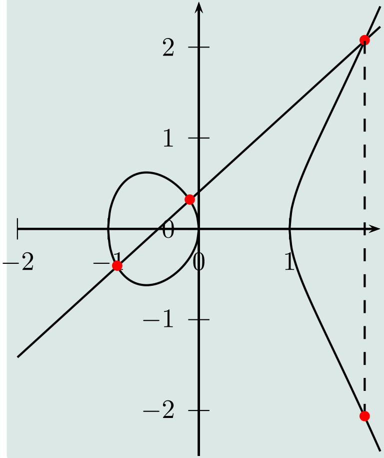

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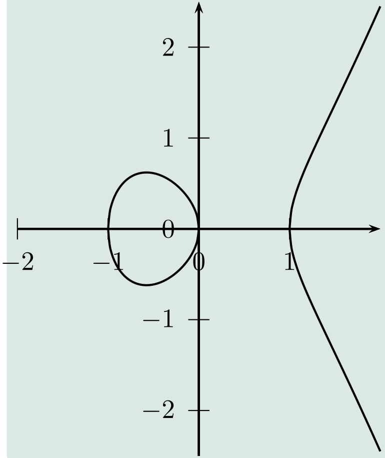

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- Double the point $P = (-0.7, 0.5975)$

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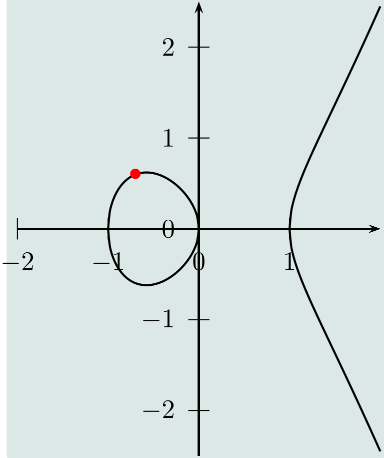

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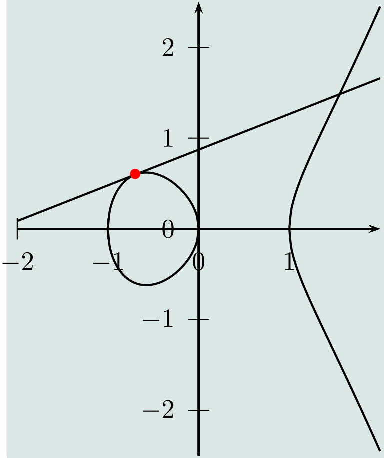

The group law

Example curve: $y^2 = x^3 - x$ over $\mathbb{R}$

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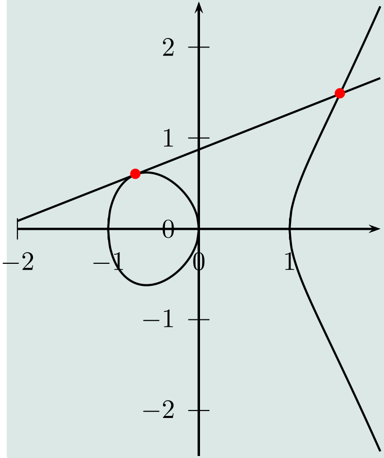

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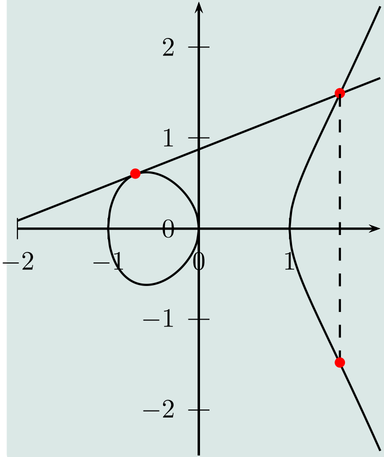

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- Note: Formulas don't work for $P + (-P)$, also don't work for $\mathcal{O}$
- Implementations need to distinguish these cases!

Finding a suitable curve

Security requirements for ECC

- $\ell = |E(\mathbb{F}_q)|$ must have large prime-order subgroup (Pohlig-Hellman)
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Finding a curve

- Fix finite field $\mathbb{F}_q$ of suitable size
- Fix curve parameter a (quite common: $a = -3$)
- Pick curve parameter b until E fulfills desired properties
- This requires efficient "point counting"
- This requires efficient factorization or primality proving

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- Various standardized curves, most well-known: NIST curves:
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- Represent points in *projective coordinates*: $P = (X_P : Y_P : Z_P)$ with $x_P = X_P/Z_P$ and $y_P = Y_P/Z_P$
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- Important: Never *send* projective representation, always convert to affine!

Problem II: group-law special cases

- Addition of $P + Q$ needs to distinguish different cases:
  - If $P = \mathcal{O}$ return Q
  - Else if $Q = \mathcal{O}$ return P
  - Else if $P = Q$ call doubling routine
  - Else if $P = -Q$ return $\mathcal{O}$
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  - If $P=\mathcal{O}$ return Q
  - Else if $Q=\mathcal{O}$ return P
  - Else if $P=Q$ call doubling routine
  - Else if $P=-Q$ return $\mathcal{O}$
  - Else use addition formulas
- Similar for doubling P:
  - If $P=\mathcal{O}$ return P
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- Constant-time implementations of this are horrible
- Good news: Can avoid the checks when computing $k \cdot P$ and $k < |E(\mathbb{F}_q)|$
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- Baseline: *simple* implementations are likely to be wrong or insecure

Solution I: Montgomery curves

- Consider elliptic curves of the form $By^2 = x^3 + Ax^2 + x$.
- Montgomery in 1987 showed how to perform x-coordinate-based arithmetic:
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- Use to efficiently compute the x-coordinate of kP given only the x-coordinate of P
- For this, let's use projective representation $(X : Z)$ with $x = (X/Z)$

One Montgomery “ladder step”

const $a24 = (A+2)/4$ (A from the curve equation)
function LADDERSTEP($x_{Q-P}, X_P, Z_P, X_Q, Z_Q$)
$t_1 \leftarrow X_P + Z_P$
$t_6 \leftarrow t_1^2$
$t_2 \leftarrow X_P - Z_P$
$t_7 \leftarrow t_2^2$
$t_5 \leftarrow t_6 - t_7$
$t_3 \leftarrow X_Q + Z_Q$
$t_4 \leftarrow X_Q - Z_Q$
$t_8 \leftarrow t_4 \cdot t_1$
$t_9 \leftarrow t_3 \cdot t_2$
$X_{P+Q} \leftarrow (t_8 + t_9)^2$
$Z_{P+Q} \leftarrow x_{Q-P} \cdot (t_8 - t_9)^2$
$X_{2P} \leftarrow t_6 \cdot t_7$
$Z_{2P} \leftarrow t_5 \cdot (t_7 + a24 \cdot t_5)$
return ($X_{2P}, Z_{2P}, X_{P+Q}, Z_{P+Q}$)
end function

The Montgomery ladder

Require: A scalar $0 \leq k \in \mathbb{Z}$ and the x-coordinate x_P of some point P
Ensure: (X_{kP}, Z_{kP}) fulfilling $x_{kP} = X_{kP}/Z_{kP}$

$x_1 = x_P$; $X_2 = 1$; $Z_2 = 0$; $X_3 = x_P$; $Z_3 = 1$
for $i \leftarrow n-1$ downto 0 **do**
  if bit i of k is 1 **then**
    $(X_3, Z_3, X_2, Z_2) \leftarrow$ LADDERSTEP$(x_1, X_3, Z_3, X_2, Z_2)$
  else
    $(X_2, Z_2, X_3, Z_3) \leftarrow$ LADDERSTEP$(x_1, X_2, Z_2, X_3, Z_3)$
  end if
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for $i \leftarrow n-1$ downto 0 **do**
  $b \leftarrow$ bit i of s
  $c \leftarrow b \oplus p$
  $p \leftarrow b$
  $(X_2, X_3) \leftarrow$ CSWAP(X_2, X_3, c)
  $(Z_2, Z_3) \leftarrow$ CSWAP(Z_2, Z_3, c)
  $(X_2, Z_2, X_3, Z_3) \leftarrow$ LADDERSTEP$(x_1, X_2, Z_2, X_3, Z_3)$
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- We only get the x coordinate of the result, tricky for signatures
- Can reconstruct y, but that involves some additional cost

Solution II: (twisted) Edwards curves

- Edwards, 2007: New form for elliptic curves ("Edwards curves")
- Bernstein, Lange, 2007: very fast addition and doubling on these curves
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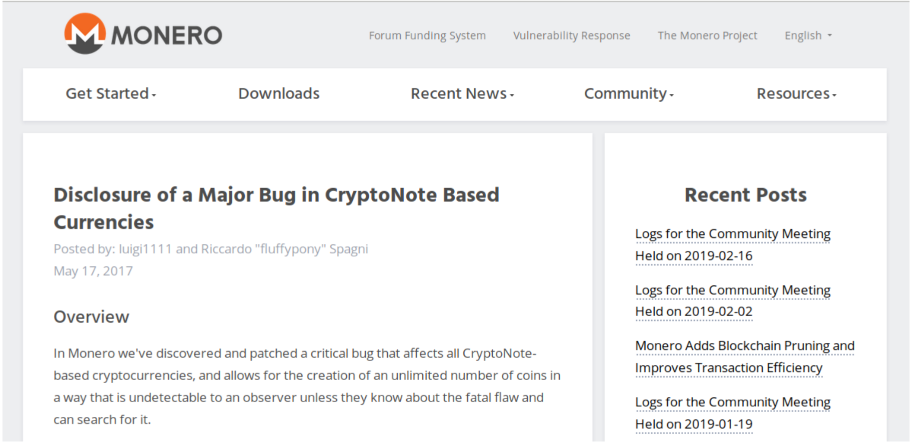
So, what's the deal with the cofactor?



MONERO

Forum Funding System | Vulnerability Response | The Monero Project | English

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Disclosure of a Major Bug in CryptoNote Based Currencies

Posted by: luigi1111 and Riccardo "fluffypony" Spagni

May 17, 2017

Overview

In Monero we've discovered and patched a critical bug that affects all CryptoNote-based cryptocurrencies, and allows for the creation of an unlimited number of coins in a way that is undetectable to an observer unless they know about the fatal flaw and can search for it.

Recent Posts

- Logs for the Community Meeting Held on 2019-02-16
- Logs for the Community Meeting Held on 2019-02-02
- Monero Adds Blockchain Pruning and Improves Transaction Efficiency
- Logs for the Community Meeting Held on 2019-01-19

So, what's the deal with the cofactor?

- Protocols need to be careful to avoid subgroup attacks
- Monero screwed this up, which allowed double-spending
- Elegant solution: "Decaf" and "Ristretto" encoding by Hamburg, see:
  - `https://eprint.iacr.org/2015/673.pdf`
  - `https://ristretto.group`
  - `https://github.com/otrv4/libgoldilocks`
- This is also used in the code of `assignment2-ecdh25519`

Solution III: Complete group law on Weierstrass curves

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- Problem: Extremely inefficient
- Renes, Costello, Batina, 2016: Much faster complete group law for Weierstrass curves
- Less efficient than (twisted) Edwards
- Overhead quite architecture-dependent (Schwabe, Sprenkels, 2019)
- Covers all curves

Problem III: Wrong-curve attacks

ECDH attack scenario

- Alice sends point on different (insecure) curve with small subgroup
- Bob computes “shared key” in that small subgroup
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- Send compressed points $(x, \text{parity}(y))$; decompression returns (x, y) on the curve or fails
- Send only x (Montgomery ladder); but: x could still be on the "twist" of E
- Make sure that the twist is also secure ("twist security")

Problem IV: Backdoors in standards?

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- Fact: There is no proof that there are no intentional vulnerabilities in NIST curves
- Question for ECC: who do you trust to pick the curve?

Point representation and arithmetic

Collection of elliptic-curve shapes, point representations and group-operation formulas by Bernstein and Lange:

https://www.hyperelliptic.org/EFD/

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- If you can choose encoding for twisted Edwards points, use Decaf/Ristretto
- Most common Montgomery / twisted Edwards curve: Curve25519
  - Defined over finite field $\mathbb{F}_{2^{255}-19}$
  - Used in Montgomery form in X25519 ECDH
  - Used in twisted Edwards form in Ed25519 signatures